On abstract 1-sections

by

Dag Normann.

1. Introduction.

In [2] , Sacks proved that every abstract 1-section is the 1-section of a normal type-two object. In the technical details of his proof he uses a selfmade hierarchy for recursion in type-two objects, and the proof is by forcing over a ramified language based on this hierarchy. In this paper we give a new proof of the same theorem without using his hierarchy. We only claim some novelty in the way of presenting the proof of the theorem.

2. Preliminaries.

We code every element in HC (the set of heriditary countable sets) as subsets of ω by defining the following partial function $h : {}^{\omega}2 \to HC$:

If $A \subseteq \omega$, then $\langle 0,A \rangle$ is a code, and $h(\langle 0,A \rangle) = A$

If $A = \{ h(X_i) : i \in \omega \}$ where $X_i \subseteq \omega$, then

$Y = \{ \langle i,j \rangle : j \in X_i \}$ is a code, and $h(Y) = A$

(We use some standard pairing function.)

The set of codes, defined by the given induction scheme, will be a complete Π^1_1-set. For further information, see § 3 of [2.]

Let $\mathfrak{N} \subseteq {}^{\omega}2$. By the structure of $\mathfrak{N}$, we mean the set of HC-sets coded by some element of $\mathfrak{N}$.

Let F be a normal type-two object. (That is a total function $F : {}^{\omega}2 \to \omega$ such that '$\exists x \in \omega$' is computable in F.) By the 1-section of F we mean the set of subsets of ω, recursive in F.

Let $M \subseteq HC$ be a transitive structure, closed under pairing and unions. We say that M satifies Δ_0-DC if : Assume $\forall x \exists y \varphi(x,y)$, where φ is a Δ_0-formula. Then there is a sequence $\langle x_i \rangle_{i \in \omega} \in M$ such that $\forall n \in \omega \ \varphi(x_n, x_{n+1})$

We say that M satisfies local countability if all elements of M can be mapped into ω by a 1-1-function in M.

M is an abstract 1-section if M is countable, M satisfies Δ_0-DC and local countibility.

We call $\mathfrak{A} \subseteq {}^{\omega}2$ an abstract 1-section if the structure of $\mathfrak{A}$ is an abstract 1-section, and $\mathfrak{A}$ is closed under recursion.

Note that if $\mathfrak{A}$ is an abstract 1-section, and M is the structure of $\mathfrak{A}$, then $\mathfrak{A} = M \cap {}^{\omega}2$.

Sacks´theorem says : A subset of ${}^{\omega}2$ is an abstract 1-section if and only if it is the 1-section of a type-two object F. To prove the if-part, use Gandys selection operator for recursion in type-two objects to prove DC. [1] Observe that calculating Δ_0-formulas is nothing more than checking quantifiers over ω .

3. Technical part.

In this part we regard an abstract 1-section to be a subset of HC.

The theorem we are going to prove is the following:

Theorem: Let M be an abstract 1-section. Then there is a $P \subset M$ such that

i) $M = L^P_{o(M)}$ and M is an abstract 1-section relative to P i.e. M satisfies $\Delta_0(P)$-DC.

ii) For all $\beta < o(M)$, L^P_β is not an abstract 1-section relative to P.

Proof: Observe that for local countable structures of the form L_α^B admissibility relative to B is equivalent to being an abstract 1-section relative to B. This is because of the definable well ordering of L_α^B, we just let X_{i+1} be the "least" element such that $\varphi(X_i, X_{i+1})$.

Let $p \in \mathbb{P}$ if L_β^p is not an abstract 1-section relative to p for any $\beta \leq rn(p)$

Let $p \leq q$ if $q = \{x \in p : rn(x) < rn(q)\}$

We get q by cutting p off at some rank.

We see that $\langle \mathbb{P}, \leq \rangle$ is Δ_1-definable and that $\langle \mathbb{P}, \leq \rangle$ is a set of conditions.

We are now going to define the forcing relation:

For Δ_0-formula $\varphi(x_1 \ldots x_n, \underline{P})$, let

$p \Vdash \varphi(x_1 \ldots x_n, \underline{P})$ if $x_1 \ldots x_n \in L^p_{rn(p)}$ and

$L^p_{rn(p)} \models \varphi(x_1 \ldots x_n, \underline{P})$, where $\underline{P}$ is interpreted as p.

For the other formulas:

$p \Vdash \neg \psi$ if $\forall q \leq p \; \neg \; q \Vdash \psi$

$p \Vdash \exists v \, \psi(v)$ if there exists an $x \in M$ such that $p \Vdash \psi(\underline{x})$

$p \Vdash \varphi \vee \psi$ if $p \Vdash \varphi$ or $p \Vdash \psi$

Note that $\Vdash^{\Delta_0}$ is Δ_1-definable.

Lemma 1: $\forall p \; \forall x \in M \; \exists q \leq p \; x \in L^q_{rn(q)}$

Proof: Let $\{x_k\}_{k \in \omega}$ be an enumeration of TC $(\{x\})$ inside M. Let $\langle i,j \rangle \in R_x \iff x_i \in x_j$. Then R_x is a wellfounded coding of x.

Let $q = p \cup \{\langle rn(p), \langle i,j \rangle \rangle : \langle i,j \rangle \in R_x\} \cup \{rn(p)+2+rn(x)\}$

The property of a relation in ω to be a code for a set is absolutely definable. Thus we may prove by induction on $rn(x_k)$

$$R_x \in L^q_{rn(p)+2}$$

$$x_k \in L^q_{rn(p)+2+rn(x_k)}$$

We must prove that $q \in \mathbb{P}$. Assume that $rn(p) < \beta \leq rn(q)$ such that L^q_β is admissible relative to q. Since for all $\gamma \leq rn(x)$ there is an element in $TC(\{x\})$ with rank γ, there is a $k \in \omega$ such that $\beta = rn(p)+2+rn(x_k)$. But then $L^q_\beta \models \forall i \exists y\ (\langle i,k \rangle \in R_x \Rightarrow R_x \restriction i$ codes y), where $\langle m,n \rangle \in R_x \restriction i$ iff $\langle m,n \rangle \in R_x$ and $\langle n,i \rangle \in R_x$. By replacement there is a v in L^q_β such that $\forall i \exists y \in v\ (\langle i,k \rangle \in R_x \Rightarrow R_x \restriction i$ codes y) and $rn(v) \geq rn(x_k)$. But then $\beta \in L^q_\beta$ $\#$.

We now see that all Δ_0-sentences will be decided, since there is a finite number of parameters in a Δ_0-formula, and Δ_0-formulas are absolute with respect to transitive structures containing the parameters, and that for generic P we will have $M = L^P_{o(M)}$.

This fact we use to prove that given a generic P, for any sentence $\varphi(x_1 \ldots x_n, \underline{P})$, we have

$$L^P_{o(M)} \models \varphi(x_1 \ldots x_n, \underline{P}) \iff (\exists p \subseteq P)(p \Vdash \varphi(x_1 \ldots x_n, \underline{P}))$$

We now prove that M is an abstract 1-section relative to P. The only thing to prove is $\Delta_0(P)$-DC, or P-replacement. For this we need

<u>Lemma 2</u>: $M \Vdash \Sigma_1$-DC

The proof is simple and we leave it as an exercise.

When we prove replacement, the main difficulty is that when $\langle p_i \rangle_{i\in\omega}$ is an increasing sequence of conditions, the $p = \bigcup_{i\in\omega} p_i$ does not need to be an element of $\mathbb{P}$. Our proof is by contradiction.

Assume that $\Delta_0^{(P)}$-replacement does not hold. Then there is a Δ_0-formula φ and a set $u \in M$ such that

$\forall x \exists y\ \varphi(x,y,P) \wedge \forall v \exists x \in u \forall y \in v \neg \varphi(x,y,P)$

Then there is a $p \in P$ such that

1) $p \Vdash \forall x \exists y\ \varphi(x,y,\underline{P})$

2) $p \Vdash \forall v \exists x \in u \forall y \in v \neg \varphi(x,y,\underline{P})$

We rewrite 1) and 2) and get

3) $\forall q \leq p \forall x \exists r \leq q \exists y\ \ r \Vdash \varphi(x,y,\underline{P})$

4) $\forall q \leq p \forall v \exists r \leq q\ r \Vdash \exists x \in u \forall y \in v \neg \varphi(x,y,\underline{P})$

From 3) we get

5) $\forall q \leq p \exists r \leq q \forall x \in L^q_{rn(q)} \exists y \in L^r_{rn(r)}\ r \Vdash \varphi(x,y,\underline{P})$

This is done by the following process: Wellorder $L^q_{rn(q)}$ of type ω and let $r_0 \leq q$ take the wellordering inside the model. By Δ_1-DC we may pick a sequence $\langle r_i \rangle_{i\in\omega}$ such that if $r = \bigcup_{i\in\omega} r_i$ we have

$r_{i+1} \leq r_i$ is of minimal rank in r such that element no. i+1 in $L^q_{rn(q)}$, x_i, gets an associated y_i such that

$r_{i+1} \Vdash \varphi(x_i,y_i,\underline{P})$

Here r_i is redefinable from r as a Δ_1-function of i, and r must be a condition, since rn(r) is not r-admissible.

Now wellorder $u = \{x_i : i \in \omega\}$ inside M, and use 5) and Δ_1-DC to get a sequence

$\langle q_i, y_i \rangle_{i \in \omega}$ such that

i) $q_i \le p$, $u \in L^{q_o}_{rn(q_o)}$

ii) $q_{i+1} \le q_i$

iii) $\forall x \in L^{q_i}_{rn(q_i)} \; \exists y \in L^{q_{i+1}}_{rn(q_{i+1})} \quad q_{i+1} \Vdash \varphi(x,y,\underline{P})$

iv) $\forall i \in \omega \quad q_{i+1} \Vdash \varphi(x_i, y_i, \underline{P})$

Let $q = \bigcup_{i \in \omega} q_i$. We will prove that $L^q_{rn(q)}$ is not an abstract 1-section relative to q, i.e. q is a condition.

$L^q_{rn(q)} \models \forall x \exists y \; \varphi(x,y)$ by iii .

Suppose $\exists v \in L^q_{rn(q)}$ such that $\forall i \in \omega \exists y \in v \; \varphi(x_i, y)$.

There is a $k \in \omega$ such that $v \in L^{q_k}_{rn(q_k)}$, since $q = \bigcup_{k \in \omega} q_k$

Then $q_k \Vdash \forall x \in u \exists y \in v \; \varphi(x,y,\underline{P})$. This contradicts 4), and $L^q_{rn(q)}$ cannot satisfy P-replacement.

Now we are ready to finish our proof.

Let $s \le q$ be such that $\langle y_i \rangle_{i \in \omega} \in L^s_{rn(s)}$, and such that $v = \{y_i ; \; i \in \omega\} \in L^s_{rn(s)}$.

Then $L^s_{rn(s)} \models \forall x \in u \exists y \in v \; \varphi(x,y,\underline{P})$. This,again, contradicts 4), and the theorem is proved.

4. Back to Sacks.

We will now prove Sacks' theorem from our.
Let $\mathfrak{A}$ be an abstract 1-section, M the structure of $\mathfrak{A}$ and P generic as in our proof. Define the function F by

$$F(X) = \begin{cases} 0 \text{ if } X \text{ is a code and } h(X) \in P \\ 1 \text{ o.w.} \end{cases}$$

We claim that $\mathfrak{A}$ is the 1-section of F, ^2E

($^2E(X) = \begin{cases} 0 \text{ if } X = \emptyset \\ 1 \text{ o.w.} \end{cases}$ $\quad$ ^2E is computing quantifiers)

1. 1-scF $\subseteq$ M

Since M is P-admissible and F is P-definable, recursion in F and ^2E can be carried out inside M.

2. $M \cap {}^{\omega}2$ = 1-scF

The structure of 1-scF is an abstract 1-section relative to P. Since M is the least such, M = structure of 1-scF. Then $M \cap {}^{\omega}2$ = 1-scF.

1. R.Gandy: General recursive functionals of finite type and hierarchies of functions. University of Clermont-Ferrand. 1962.

2. G.E.Sacks: The 1-section of a type n object. Proceedings from the Oslo-symposium on recursion theory. 1972. To appear on North-Holland.